

A toy model for Macroscopic Quantum Coherence

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Abstract

The present article deals with Macroscopic Quantum Coherence resorting only to basic quantum mechanics. A square double well is used to illustrate the Leggett-Caldeira oscillations. The effect of thermal-radiation on two-level systems is discussed to some extent. The concept of decoherence is introduced at an elementary level. Handles are deduced for the energy, temperature and time scales involved in Macroscopic Quantum Coherence.

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I. INTRODUCTION

Triggered by a seminal article¹ written by A J Leggett in 1980, research into Macroscopic Quantum Coherence (MQC) has yielded impressive experimental,^{2–5} theoretical^{6–9} and even technological achievements^{10–12}. The ideas developed in the last thirty so years by Leggett and his collaborators have not only changed the way we understand the relation between quantum and classical behaviours, but are also crucial in the future development of quantum computing. The present article aims at explaining the basic phenomenology of MQC resorting only to basic quantum mechanics. Thus, we believe this article can be of interest for any student who has attended at least a one-year course in quantum physics, and for faculty members committed to introducing students into contemporary research.

In order to explain briefly what MQC is, let us consider a particle in a symmetric double well potential (SDWP). Figure 1 depicts an example of such a potential. In freshmen courses we have been told what to expect when the particle is in a high-lying energy level in a nice, analytical, potential such as this: for states for which the change in potential energy within a de Broglie wavelength is much smaller than the mean kinetic energy, the specifically quantum features of the behavior result negligible and the classical description becomes adequate.¹³ In that sense, classical behaviour can be considered as a limiting case of quantum mechanics¹⁴. Suppose, nonetheless, the central barrier in the SDWP of Fig. 1 to be of macroscopic width. Then, the predictions of quantum mechanics and classical mechanics certainly clash for this system. A classical viewpoint would demand two distinct localized states of stable equilibrium, situated at $-x_0$ and x_0 , while quantum mechanics predicts an even probability distribution for the (non-degenerate¹⁵) ground level state (which is, of course, the more stable stationary state.) In fact, the ground statefunction for such a potential would have to look something like Figure 2.

Moreover, the (odd) eigenfunction of the first excited level (Figure 3), and indeed each one of the stationary solutions of a SDWP, necessarily has an even probability distribution.

What Leggett predicted more than thirty years ago, and what actually happens in experiments carried out in SDWPs of micrometric and nanometric typical lengths, is the appearance of a two-fold degenerate ground level E' , with the system oscillating in an harmonic fashion between two eigenstates, $|L\rangle$ (Figure 4) and $|R\rangle$ (Figure 5) localized, respectively,

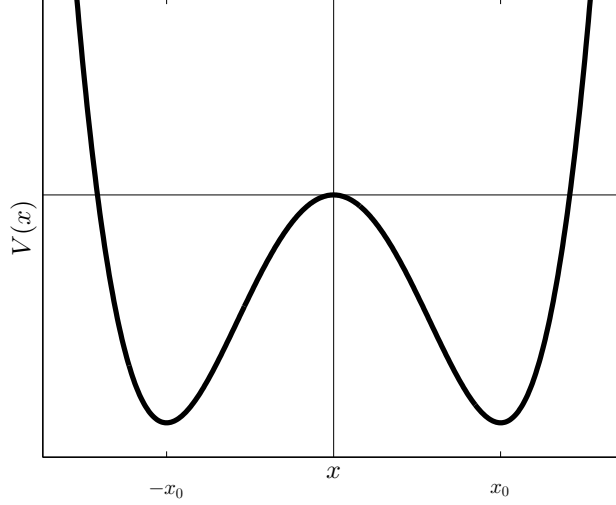


FIG. 1. An example of a SDWP potential, with characteristic double minima and central peak.

at the left and right of the central barrier. At ground level, the position expectancy value oscillates in the accordance with:

$$\langle x \rangle_0(t) = \langle x \rangle_0(0) \cos \omega t. \quad (1)$$

This phenomenon, the so called Leggett-Caldeira oscillations, is closely related with the Rabi oscillations of atomic physics. It is explained as the result of the purported ground level E' resolving into a true ground level

$$E_+ = E' - \hbar\omega/2, \quad (2)$$

endowed with an even non-localized eigensolution $|+\rangle$, and a first excited level

$$E_- = E' + \hbar\omega/2, \quad (3)$$

endowed with an odd non-localized eigensolution $|-\rangle$. When a quantum system tunnels periodically trough the barrier of a SDWP with a central barrier of macroscopic length, we have Macroscopic Quantum Coherence.

The states $|R\rangle$ and $|L\rangle$ have, each one on its own, a definite value of a macroscopic property (namely, the property of being localized at the left or the right of the barrier). At the same time, $|R\rangle$ and $|L\rangle$ are linear combinations of the states $|+\rangle$ and $|-\rangle$, which cannot be said to be localized. In order to understand Leggett's original motivation, notice the analogy between macroscopic SDWPs and Schroedinger's cat: the celebrated pet can

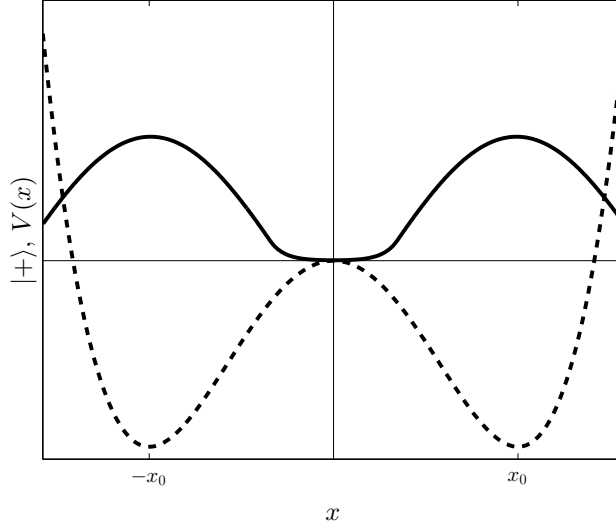


FIG. 2. A rendering of what a ground-level eigenfunction (solid curve) should look like for a SDWP. The potential is shown as a dashed curve.

be in any of two different *macroscopically distinguishable* states (let us say, Ψ_1 for a live cat and Ψ_0 for a dead one) just as a particle in a SDWP. If any of these macroscopic systems obeys the laws of quantum mechanics, then it could be prepared in linear combinations that lack a sharp, well defined, value of the macroscopic property. Examples of these linear combinations are the $|\pm\rangle$ states of the SDWPs, and the “neither dead nor alive” states

$$\Psi_{\pm} = \frac{1}{\sqrt{2}}(\Psi_0 \pm \Psi_1) \quad (4)$$

of the cat. Thus, a more general definition of MQC is simply: the quantum superposition of distinct macroscopic states. Long time before the year of 1980, macroscopic quantum phenomena had been discovered: superconductivity in 1911, and superfluidity in 1937. Yet it remained for Leggett to identify the conditions necessary for a quantum system to present macroscopically distinguishable states.¹

Some twenty years elapsed between Leggett’s proposal and a credible experimental confirmation^{4,5} of MQC. One of the main reasons for this delay lies in the fact that the phase coherence of the $|\pm\rangle$ states is rapidly lost due to the interaction of the system with its surroundings, so the system collapses into one of the localized states before one period of the Leggett-Caldeira oscillation is completed.^{1,8}

MQC is relevant not only from the purely theoretical point of view. A physical qubit is a two-level system considered as a piece of hardware. And, as we shall see in the following

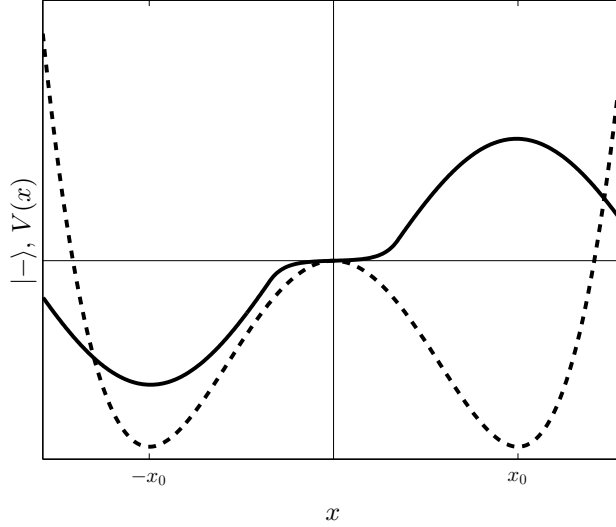


FIG. 3. The first excited level eigenfunction (solid) of a SDWP (dashed).

pages, at least some SDWPs can behave as effective two-level systems at sufficiently low temperatures. Quantum computing (an area with impressive software development, but little hardware to show) requires qubits to interact with one another without loss of coherence, for fairly long times, even at fairly high temperatures. Thus, the study of two-level dissipative systems, to which Leggett and collaborators made far reaching contributions when delving in the foundations of quantum physics, has revealed itself crucial for people in the vanguard of technological development.^{3,5}

The rest of this article is structured as follows: in section II we discuss the spectra of a family of symmetric double square well potentials, and the conditions under which a member of this family can be considered as an effective two-level system. Next, the properties of two-states systems arising from SDWPs are discussed in section III. We then go on to examine in IV how thermal radiation, by throwing the system into higher energy levels, renders the two-level model inapplicable. In section V decoherence is introduced in elementary terms, and its relation with dissipation is briefly discussed. Handles for the time, energy and temperature scales involved in MQC are derived from our toy model in section VI. Finally, conclusions are laid down in section VII.

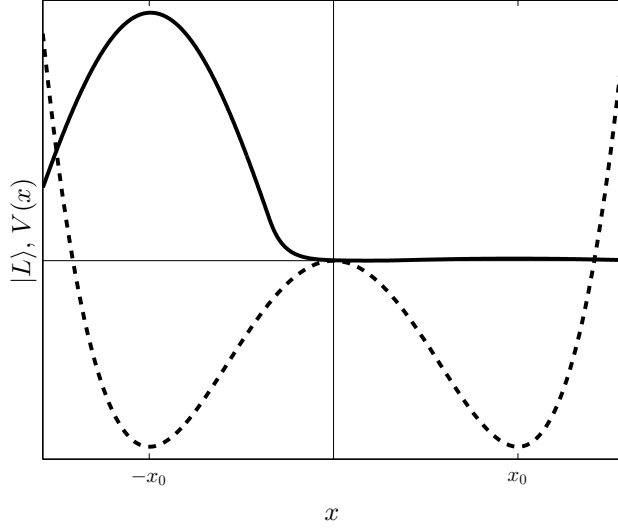


FIG. 4. The $\langle x|L \rangle$ state (shown solid) localized at the left of the SDWP (shown dashed) barrier.

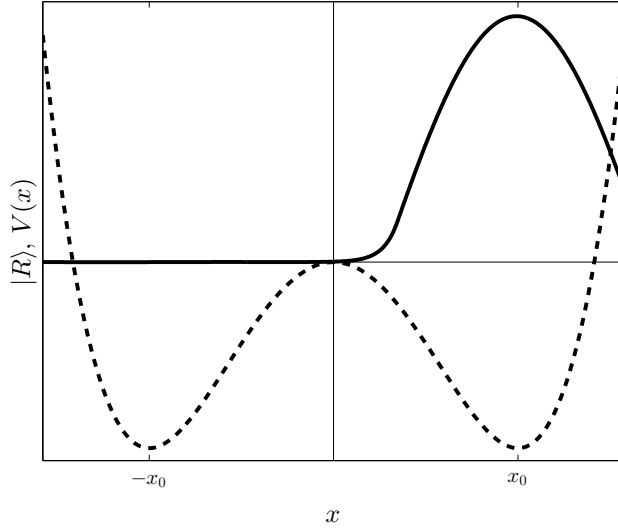


FIG. 5. The $\langle x|R \rangle$ state (shown solid) localized at the right of the SDWP (shown dashed) barrier.

II. SYMMETRIC DOUBLE SQUARE WELLS

Leggett resorted to quasi-classical considerations when stating his original proposal.¹ Also, the WKB approximation has been applied to double well potentials by Landau and Lifshitz,¹⁶ and more recently, in this Journal, by others.¹⁷ Here will take a different point of view, avoiding all together quasi-classical approximations, by considering a particular family of double infinite square well potentials as approximations to actual, analytic SDWPs. Our

procedure will later allow us to get some reference values on the energies, temperatures and times involved in MQC. The following family of piece-wise-constant potentials will be considered:

$$U_b(x) = \begin{cases} \infty & \text{if } x \leq -a - b, \\ 0 & \text{if } -b > x > -a - b, \\ k & \text{if } b \geq x \geq -b, \\ 0 & \text{if } b + a > x > b, \\ \infty & \text{if } x \geq b + a. \end{cases} \quad (5)$$

Potentials of this kind have previously been studied in a different context, and it has been shown¹⁸ that, if all other parameters held fixed, levels E_{2n+1} and E_{2n} coalesce as $k \rightarrow \infty$. Here we shall consider the barrier height $k > 0$ as a fixed number, although “big” in a sense that will be readily clarified. This, in order to keep the gap between the ground and first excited levels sufficiently small. We shall also take the width of each one of the lateral valleys, $a > 0$, as a fixed value unless otherwise stated, leaving free the only other parameter, that is the barrier half-width $b > 0$.

One of the two main objectives of this Section is to obtain a global lower bound on the energy gap between the first and second excited levels in the U_b potentials. Just as important to our ends, we will learn on this Section that there is a “running” upper bound (dependent on the value of b) on the gap between the ground and first excited levels. The consequences of this to facts, which are vital to the rest of the article, are explored in Sections III, IV and VI.

To be sure, non of the U_b is continuous, yet they share the most prominent features of a SDWP, namely, they are even potentials with completely bounded, non-degenerate, spectra, as can be shown from boundary conditions. If instead of two minima, the U_b have two non-overlapping regions of minima (*viz.* $(-a - b, -b)$ and $(b, a + b)$), this distinction will prove to be quite unimportant.

Also from boundary conditions (or from more abstract, symmetry considerations) it is readily seen that the levels in the spectrum of any of the U_b are classified according with parity, just like it happens for a continuous even potential:

$$\psi_{2n,b}(-x) = \psi_{2n,b}(x), \quad n = 0, 1, \dots, \quad b \in (0, \infty) \quad (6)$$

and

$$\psi_{2n+1,b}(-x) = -\psi_{2n+1,b}(x), \quad n = 0, 1, \dots, \quad b \in (0, \infty). \quad (7)$$

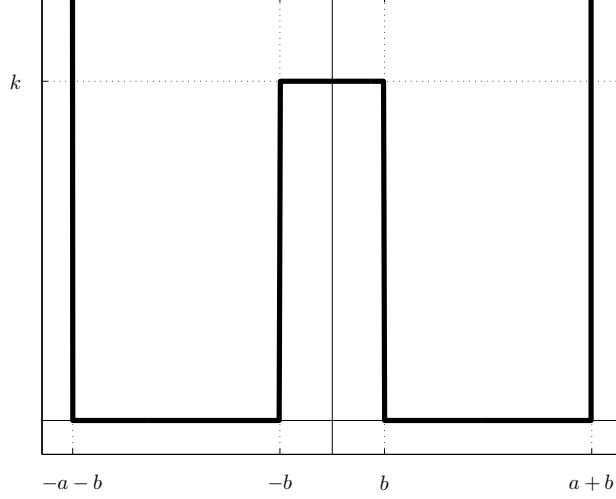


FIG. 6. A typical member of the $U_b(x)$ family of potentials

Let us focus on the discretization conditions below the level of the central barrier ($E < k$). From the boundary conditions we get, for even states:

$$-\sqrt{E_{2n}} \cot a \frac{\sqrt{2mE_{2n}}}{\hbar} = \sqrt{k - E_{2n}} \tanh b \frac{\sqrt{2m(k - E_{2n})}}{\hbar}, \quad (8)$$

while odd levels below the barrier level have to comply with

$$-\sqrt{E_{2n+1}} \cot a \frac{\sqrt{2mE_{2n+1}}}{\hbar} = \sqrt{k - E_{2n+1}} \coth b \frac{\sqrt{2m(k - E_{2n+1})}}{\hbar}. \quad (9)$$

Notice how the first of these two conditions can be written in the form:

$$g(E_{2n}) = h_b(E_{2n}), \quad (10)$$

and the second can be rendered as:

$$g(E_{2n+1}) = j_b(E_{2n+1}), \quad (11)$$

with the meaning of g , h_b and j_b being obvious from the context.

Both of these two last equations are depicted in Figure 7, from which it can be seen that there exists an upper bound B , given by

$$B = \frac{\pi^2 \hbar^2}{2ma^2}, \quad (12)$$

such that the ground and first excited states have to comply with

$$\frac{B}{4} < E_0 < E_1 < B, \quad (13)$$

no matter the value of b . Obviously, there can be no levels below the barrier unless $k > B/4$. We shall only consider potentials for which the condition:

$$k \gg B \quad (14)$$

is met, so that we will always have at least two levels below the barrier. Indeed, the number of levels below the barrier increases with increasing quotient k/B and, more importantly, as B is independent of k , condition (14) warrants that the gap between the first two level is always small. It is not difficult to generalize (13) starting from (8) and (9) and definition (12). The result is that:

$$(n + \frac{1}{2})^2 B < E_{2n,k} < E_{2n+1,k} < (n + 1)^2 B, \quad n = 0, 1, 2, \dots, N, \quad (15)$$

if the level $2N + 1$ is still below the barrier.

From inequality (15) it follows that

$$E_{2n+2} - E_{2n+1} > (n + 5/4)B, \quad n = 0, 1, 2, \dots, N \quad (16)$$

if level $2N + 1$ is below the barrier. We then have that the gap between the ground and first excited levels will always be less than the gap between the first and second excited levels:

$$E_2 - E_1 > \frac{5}{4}B > \frac{3}{4}B > E_1 - E_0. \quad (17)$$

But we can do much more better than that. Indeed, in Appendix A it is formally proven that for any given number $\delta > 0$ there exist a value $b(\delta) > 0$ such the gap between the ground and the first excited level of a U_b potential will be less than δ , that is

$$E_1 - E_0 < \delta, \quad (18)$$

if $b \geq b(\delta)$. In other words, if we choose the barrier length big enough, then we can make E_0 and E_1 as proximate as we want, while there is a lower bound for the gap between E_2 and E_1 which is independent of the value of this length. This will allow us to find examples of U_b that will work as effective two-state systems for the lowest-lying energy levels, as illustrated in Figure 8.

Finally, there is one more inequality that can be derived from (15) and that will prove useful in section IV. This inequality is:

$$E_2 - E_1 < \frac{15}{4}B. \quad (19)$$

Let us stress that relations (13), (17) and (19) are verified for each U_b regardless of the value of b .

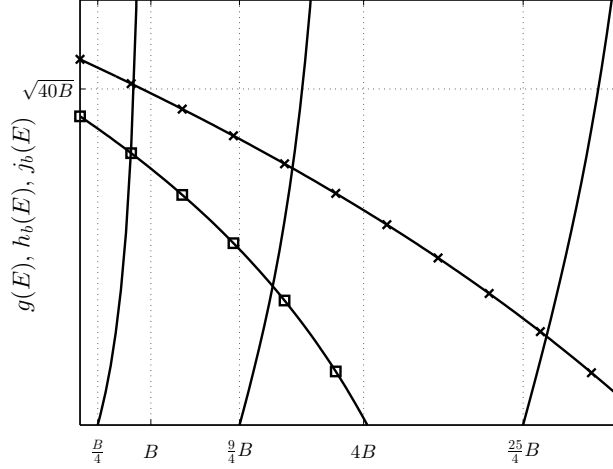


FIG. 7. Graphical solutions of transcendental equations (8) and (9). Depicted, functions $g(E)$ (solid), $h_b(E)$ (squares) and $j_b(E)$ (crosses). In all cases $k = 40B$ and $b = 0.8a$. In this example only the ground level and the three first excited levels are below the barrier height k .

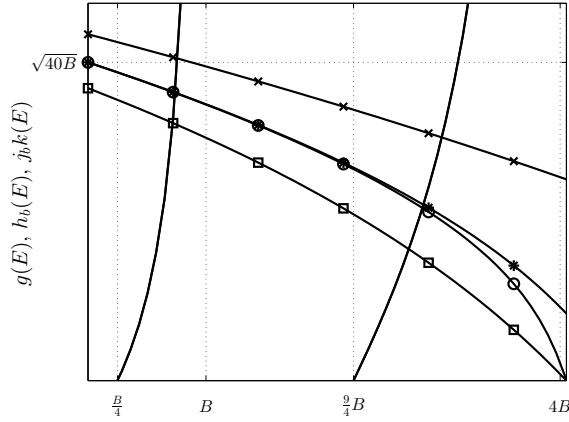


FIG. 8. Doubling the barrier width produces a dramatic decrease in the gap between the ground and first excited levels. Shown: function h_b for $b = 0.8a$ (squares) and $b = 0.4a$ (circles), and function j_b for $b = 0.8a$ (crosses) and $b = 0.4a$ (asterisks). In all cases $k = 40B$.

III. TWO-LEVEL SYSTEMS WITH REFLECTION SYMMETRY

In the preceding section we have proven that there are U_b potentials for which the gap between the ground and first excited energy levels is much more narrow than the one between

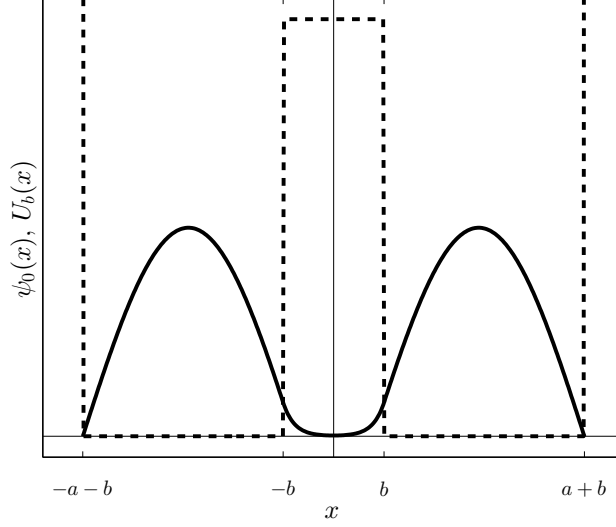


FIG. 9. Solid curve: the $\psi_+(x)$ ground-level eigenfunction (even), obtained through computer assisted numerical analysis. Dashed: the corresponding U_b potential. In this example $k = 14B$ and $b = 0$.

the first and second excited levels. Consequently, for low energy expectancy values, a particle in one of such potentials acts as an effective two-level systems.^{19,20}

In the rest of this section we shall consider a fixed U_b that behaves as a two-level system, and drop the b .

Consider now the non-stationary solutions ψ_L and ψ_R that one obtains from the linear combinations

$$\psi_L(x, t) = \frac{1}{\sqrt{2}} \left[\exp \left(-i \frac{E_0 t}{\hbar} \right) \psi_0(x) + \exp \left(-i \frac{E_1 t}{\hbar} \right) \psi_1(x) \right] \quad (20)$$

and

$$\psi_R(x, t) = \frac{1}{\sqrt{2}} \left[\exp \left(-i \frac{E_0 t}{\hbar} \right) \psi_0(x) - \exp \left(-i \frac{E_1 t}{\hbar} \right) \psi_1(x) \right]. \quad (21)$$

These states have no definite parity, but instead one is the specular image of the other:

$$\psi_L(-x, t) = \psi_R(x, t), \quad (22)$$

as can be seen from equations (6), (7), (20) and (21).

The position expectancy value for this states is calculated from (6) in a straightforward manner:

$$\langle x \rangle_L(t) = -\langle x \rangle_R(t) = \langle \psi_0 | x | \psi_1 \rangle \cos \frac{E_1 - E_0}{\hbar} t, \quad (23)$$

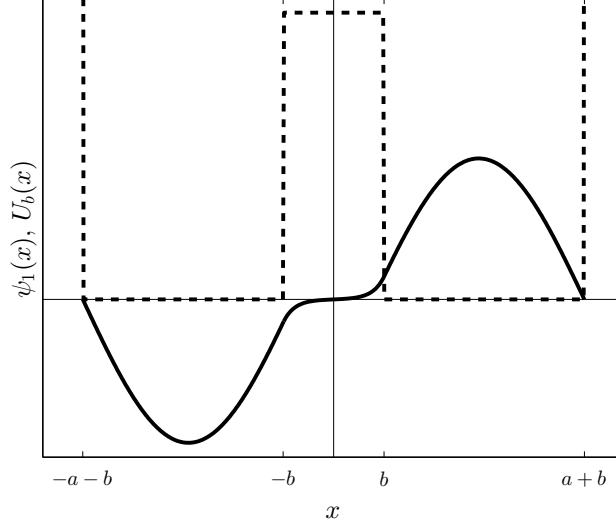


FIG. 10. Solid: the (odd) eigenfunction of the first excited level. Dashed: the U_b potential to which this solution corresponds. In this example $k = 14B$ and $b = 0.2a$.

as is the energy expectancy value:

$$\langle H \rangle_L = \langle H \rangle_R = \frac{E_0 + E_1}{2}. \quad (24)$$

Comparing (23) with (1) and (24) with (2) one may be tempted to make the identifications

$$E' = \langle H \rangle_L \quad \text{and} \quad \omega = \frac{E_1 - E_2}{\hbar}, \quad (25)$$

from which (3) would follow, so that the states of (20) and (21) could be interpreted as the localized states observed in the experiments, and ψ_0 and ψ_1 would correspond to the true ground level E_+ and the first excited state E_- . That is, it would be cogent that

$$\langle x|L \rangle = \psi_L(x), \quad \langle x|R \rangle = \psi_R(x), \quad \langle x|+ \rangle = \psi_0(x), \quad \langle x|- \rangle = \psi_1(x). \quad (26)$$

In this interpretation, however, there is no room for transitions. Indeed, the complete Schroedinger equation for a U potential, which reads:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(x, t) + U(x)\psi(x, t) = i\hbar \frac{\partial \psi}{\partial t}(x, t), \quad (27)$$

predicts that if the system is initially prepared in the state $\psi_L(x, t = 0)$ at time $t = 0$, then it will remain in the $\psi_L(x, t)$ state for $t \in [0, \infty)$ (which is a sophisticated way to say: *forever*). This is just consequence of PDE's theory.

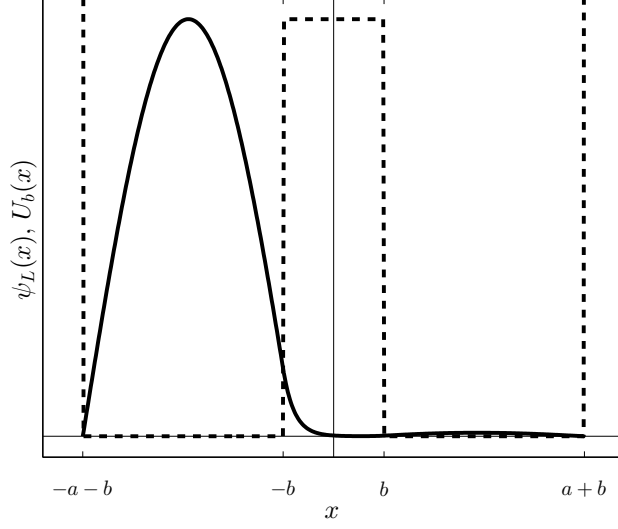


FIG. 11. The localized $\psi_L(x, t = 0)$ eigenfunction (solid) of the U_b potential (dashed) for $b = 0.4a$ and $k = 14B$.

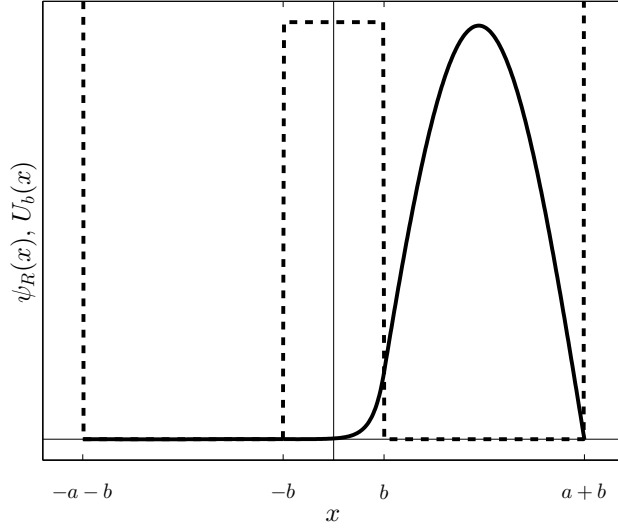


FIG. 12. The localized $\psi_R(x, t = 0)$ eigenfunction (solid) of the U_b potential (dashed) for $b = 0.4a$ and $k = 14B$.

Instead of periodical transitions between two different states, our equations predict the existence of a unique “oscillating” state, because $\psi_R(x, t)$ is a time-displaced replica of $\psi_L(x, t)$:

$$\psi_L\left(x, t + \frac{\pi}{\omega}\right) = i \exp\left(-i \frac{\pi \Omega}{\omega}\right) \psi_R(x, t) \quad , \quad (28)$$

where Ω stands for

$$\Omega = (E_1 + E_2)/2\hbar \quad (29)$$

and ω is as in (25). One arrives at this result directly from (20) and (21) after some algebra.

A. Flip-flops and Leggett-Caldeira oscillations

Let us start from what we know happens in actual experiments (*i. e.* the existence of an observable degenerate ground level) and proceed to deduce from there the perturbation needed to achieve such degeneracy. The SDWP Hamiltonian H is represented by the matrix

$$\mathbb{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} \quad (30)$$

in the symmetry-respecting basis formed by the eigenfunctions ψ_0 and ψ_1 . Let us consider another Hamiltonian, H' , represented by the matrix

$$\tilde{\mathbb{H}}' = \mathbb{O}\mathbb{H}'\mathbb{O}^{-1} = \begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix} \quad (31)$$

in the symmetry-violating basis spanned by ψ_L and ψ_R . Here, \mathbb{O} stands for the unitary operator which transforms ψ_0 into ψ_L and ψ_1 into ψ_R , thus:

$$\mathbb{O} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbb{O} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (32)$$

Notice that, as the rhs of (31) is proportional to the identity matrix, then:

$$\mathbb{H}' = \begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix}. \quad (33)$$

Thus, the perturbation is represented by:

$$\mathbb{W} = \mathbb{H}' - \mathbb{H} = \begin{pmatrix} \hbar\omega/2 & 0 \\ 0 & -\hbar\omega/2 \end{pmatrix}. \quad (34)$$

Now, in the basis spanned by ψ_L and ψ_R the things look quite different. Indeed, we have that:

$$\tilde{\mathbb{H}} = \mathbb{O}\mathbb{H}\mathbb{O}^{-1} = \begin{pmatrix} E' & -\hbar\omega/2 \\ -\hbar\omega/2 & E' \end{pmatrix} \quad (35)$$

and most importantly:

$$\tilde{\mathbb{W}} = \mathbb{O}\mathbb{W}\mathbb{O}^{-1} = \begin{pmatrix} 0 & \hbar\omega/2 \\ \hbar\omega/2 & 0 \end{pmatrix}. \quad (36)$$

So that the perturbation has no diagonal elements. This means that zeroth order corrections are strictly null for the perturbation, and, further more, that the off-diagonal elements are equal.

To be very clear, let us write the eigen-equations for each one of this distinct systems. For H we have

$$H\psi_0(x, t) = i\hbar\frac{\partial\psi_0}{\partial t}(x, t) = E_0\psi_0(x, t), \quad H\psi_1(x, t) = i\hbar\frac{\partial\psi_1}{\partial t}(x, t) = E_1\psi_1(x, t), \quad (37)$$

while H' responds to

$$H'\psi_L(x, t) = i\hbar\frac{\partial\psi_L}{\partial t}(x, t) = E'\psi_L(x, t), \quad H'\psi_R(x, t) = i\hbar\frac{\partial\psi_R}{\partial t}(x, t) = E'\psi_R(x, t). \quad (38)$$

Now, let us consider $\tilde{\mathbb{H}}'$ as the initial, unperturbed, Hamiltonian matrix, and

$$- \tilde{\mathbb{W}} = -\mathbb{O}\mathbb{W}\mathbb{O}^{-1} \quad (39)$$

as the perturbation, so that $\tilde{\mathbb{H}}$ is the final, perturbed, Hamiltonian matrix. Then we can show that the $\psi_L(x, t)$ and $\psi_R(x, t)$ states transit from one another in Rabi style. Indeed, resorting to the time-dependent perturbation formalism,^{22,23} we write, for a general state $\psi(x, t)$ of $\tilde{\mathbb{H}}$,

$$\psi(x, t) = c_L(t)\psi_L(x, t) + c_R(t)\psi_R(x, t), \quad (40)$$

in order to obtain the equation

$$i\hbar\frac{d}{dt} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (t) = \tilde{\mathbb{W}} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (t), \quad (41)$$

which is equivalent to the 2×2 system of coupled linear equations:

$$i\hbar\frac{dc_L}{dt} = -\frac{\hbar\omega}{2}c_R, \quad i\hbar\frac{dc_R}{dt} = -\frac{\hbar\omega}{2}c_L. \quad (42)$$

By uncoupling this system we get the harmonic oscillator equation

$$\frac{d^2c_L}{dt^2} = -\frac{\omega^2}{4}c_L \quad (43)$$

and a similar equation for c_R , so that

$$c_L(t) = \sin(\omega t/2 + \phi) \quad c_R(t) = \cos(\omega t/2 + \phi), \quad (44)$$

where ϕ is a constant that can be elucidate from initial conditions. The probability of finding the particle in the state ψ_L is given, according to this last equations, by

$$P_L(t) = \sin^2(\omega t/2 + \phi), \quad (45)$$

and the probability of finding the the particle in the ψ_R state is

$$P_R(t) = 1 - P_L(t). \quad (46)$$

This is a particular instance of the Rabi oscillation, and this case is resonant due to the degeneracy of the “initial” Hamiltonian \tilde{H}' . But the “perturbed” Hamiltonian H is nothing else than the SDWP Hamiltonian of equation (27).

Now, equations (45) and (46) predict the “flip-flop” between the stationary states $\psi_R(x)$ and $\psi_L(x)$, so that, if the system is initial prepared in the state

$$\psi(x, t = 0) = \psi_L(x), \quad (47)$$

then we will have a 100% certainty to find it in state $\psi_R(x)$ at times $t = \frac{\pi}{\omega}, \frac{3\pi}{\omega}, \frac{5\pi}{\omega} \dots$ and a 100% certainty to find it in state $\psi_L(x)$ at times $t = \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \frac{6\pi}{\omega} \dots$. And this last result is consistent with equation (28). Thus, we are in the presence of two different (yet not contradictory) descriptions of one and the same phenomenon: if H is considered an unperturbed Hamiltonian, with complete stationary solutions $\psi_0(x, t)$ and $\psi_1(x, t)$, then we have an “oscillating” non-stationary solution $\psi_L(x, t)$. If, on the other hand, H is considered to be the result of a perturbation acting on the degenerate Hamiltonian H' , we then get flip-flops between the complete stationary solutions of H' , that is: $\psi_L(x, t)$ and $\psi_R(x, t)$.

In this manner, we obtain the periodic transitions (the zero point Leggett-Caldeira oscillations) observed in so many experiments. Notice that this transitions occur in the absence of external fields, thus without emission or absorption.

IV. THERMAL RADIATION

Due to the fact that no quantum system can be completely isolated from its environment, in any realistic description the Schroedinger equation must be supplemented with terms

that describe the interaction between the system and its surroundings. But there are very different ways to describe this interaction and its results, depending on the time and energy scales involved, and the complexity of the analysis. Here we shall discuss the absorption-induced transitions by which the system is thrown into high-lying energy levels, rendering the two-level model inapplicable. The main result from this discussion will be a limit on the temperature at which Caldeira-Leggett oscillations can be observed.

A. Oscillations near resonance

Now, oscillatory behavior is to be expected not only for the the resonant, exactly degenerate, Hamiltonian matrix \mathbb{H}' . Indeed, it would not be realistic to expect Leggett-Caldeira oscillations only in perfectly isolated systems. Consider a harmonic perturbation of the SDWP matrix Hamiltonian \mathbb{H} of equation (30), that is, a perturbative term of the general form

$$\mathbb{V} = \mathbb{A} \exp(i\omega't) + \mathbb{A}^\dagger \exp(-i\omega't), \quad (48)$$

and let us focus on the particularly simple case for which

$$\mathbb{A} = A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (49)$$

so that the perturbative term can be written down as

$$\mathbb{V} = A \begin{pmatrix} 0 & \exp(i\omega't) \\ \exp(-i\omega't) & 0 \end{pmatrix}. \quad (50)$$

We then again resort to the time-dependent perturbation formalism, and write

$$\psi(x, t) = c_0(t)\psi_0(x, t) + c_1(t)\psi_1(x, t) \quad (51)$$

in order to obtain the equation

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (t) = \mathfrak{V}(t) \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} (t), \quad (52)$$

where \mathfrak{V} , defined by

$$\mathfrak{V}(t) = \exp(i\mathbb{H}t/\hbar) \mathbb{V}(t) \exp(-i\mathbb{H}t/\hbar) \quad (53)$$

represents the perturbation in the interaction picture, and in our particularly simple case reduces to

$$\mathfrak{V}(t) = A \begin{pmatrix} 0 & \exp(i(\omega' - \omega)t) \\ \exp(-i(\omega' - \omega)t) & 0 \end{pmatrix}, \quad (54)$$

so that equation (52) is equivalent to the 2×2 system of coupled ODEs

$$i\hbar \frac{dc_0}{dt} = A \exp[i(\omega' - \omega)t] c_1, \quad i\hbar \frac{dc_1}{dt} = A \exp[-i(\omega' - \omega)t] c_0. \quad (55)$$

It can be checked by hand that

$$c_0(t) = \exp(i\Omega't/2) \left\{ \cos(R_0 t) - \frac{i\Omega'}{2R_0} \sin(R_0 t) \right\} \quad (56)$$

and

$$c_1(t) = \frac{-iR_1}{R_0} \exp(-i\Omega't/2) \sin(R_0 t) \quad (57)$$

provide a solution for the initial conditions $c_0(t=0) = 1$, $c_1(t=0) = 0$. Here we have used the following shorthand

$$R_0 = \sqrt{(A/\hbar)^2 + \left(\frac{\omega' - \omega}{2}\right)^2}, \quad \Omega' = \omega' - \omega \quad \text{and} \quad R_1 = A/\hbar, \quad (58)$$

which lead to what is known as Rabi's formula,²⁴ namely:

$$P_1(t) = \left(\frac{R_1}{R_0}\right)^2 \sin^2(R_0 t), \quad (59)$$

$$P_0(t) = 1 - P_1(t). \quad (60)$$

It is not difficult to find the expressions for $P_L(t)$ and $P_R(t)$ for this particular choice of \mathbb{A} . We omit these, as they are not particularly illuminating. Let us just point out that in all instances P_R and P_L are oscillating functions of time, although they are generally not periodic. If one wishes to describe periodic Rabi oscillations in the R and L states, one should take, instead of (49),

$$\mathbb{A} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad (61)$$

as the natural choice for \mathbb{A} . By doing this one obtains expressions completely analogous to (59) and (60) for P_L and P_R :

$$P_L(t) = \left(\frac{R_1}{R'_0}\right)^2 \sin^2(R'_0 t) \quad (62)$$

$$P_R(t) = 1 - P_L(t), \quad (63)$$

where the new frequency of the oscillation is now given by

$$R'_0 = \sqrt{(A/\hbar)^2 + (\omega'/2)^2} \quad (64)$$

The main conclusion of this subsection is thus, that the Leggett-Caldeira can survive the influence of an environment on the particle in a SDWP under certain circumstances.

B. A limit on temperature

An important result from perturbation theory tells us that for harmonic perturbations the time-dependent transition amplitude, $c_{n \rightarrow m}(t)$, between two given eigenstates of the complete Hamiltonian H is given by:²⁵

$$c_{n \rightarrow m}(t) = \langle \psi_m | A | \psi_n \rangle \frac{1 - \exp i(\frac{E_m - E_n}{\hbar} - \omega')t}{E_m - E_n - \hbar\omega'} + \langle \psi_m | A^* | \psi_n \rangle \frac{1 - \exp i(\frac{E_m - E_n}{\hbar} + \omega')t}{E_m - E_n + \hbar\omega'}. \quad (65)$$

As a consequence we get that, if the system is to stay in the two lowest lying levels, then the perturbation must meet the condition:

$$\omega' < \frac{E_2 - E_1}{\hbar}. \quad (66)$$

Otherwise, the perturbation would excite the system to higher levels with non-negligible probability. This gives a limit on the temperature at which the system behaves like a low-lying two-level system. Indeed, recalling Wien's law for blackbody radiation, we get that thermal radiation at a temperature T will have a maximal contribution of frequency ω' when condition

$$\omega' = \frac{2\pi c}{b_W} T \quad (67)$$

is met. (Here, T stands for the temperature of the radiation, b_W is Wien's constant, and c the velocity of light) Thus, if $\mathbb{V}(x, t)$ is somehow to represent thermal radiation, and if the perturbed Hamiltonian H' is to be described as a low-lying two-level system, then we must have:

$$T < \frac{b_W}{2\pi c} \frac{E_2 - E_1}{\hbar}. \quad (68)$$

In so many words: for each system there is a limit temperature above which the two-level system description is inapplicable, and zero-point Leggett-Caldeira oscillations become

overshadowed by other transitions. Moreover, from inequalities (17) and (19) we get :

$$T_B(a, m) < \frac{b_W}{2\pi c} \frac{E_2 - E_1}{\hbar} < 3T_B(a, m) \quad (69)$$

with this global bound given by:

$$T_B(a, m) = \frac{5\pi\hbar b_W}{16mca^2} \quad (70)$$

The meaning of expressions (69) and (70) is the following: consider a family of double rectangular barriers, with a fixed m , a and k , but free barrier width. When exposed to thermal radiation, there is a temperature T_B for the radiation above which the Leggett-Caldeira oscillations are overshadowed by other transitions in least some the systems, and at temperature $3T_B$ the Calderia-Legget oscillations are surpassed by other transitions in all of the systems.

V. DECOHERENCE AND DISSIPATION

We begin this section with a simplified exposition of mixed and pure states and the density matrix formalism as found in Landau and Lifshitz,²⁶ to move on next to an also simplified rendering of some of Leggett's original argumentation. After that, decoherence is defined, and its relation with dissipation is briefly discussed.

The interaction of a system (\mathfrak{S}) with its surroundings (\mathfrak{E}) can be taken into account by considering a bigger *isolated* system (\mathfrak{U}) which encompasses both \mathfrak{S} and \mathfrak{E} (that is: $\mathfrak{U} = \mathfrak{S} \cup \mathfrak{E}$). The state of this new, all including, system, \mathfrak{U} is described by a state function $\Psi(j, \xi)$ that depends both on the coordinates of \mathfrak{S} (the j) and the the coordinates of its environment (the ξ). The total Hamiltonian H_T acting on \mathfrak{U} can always be written in the form:

$$H_T = H + H_{\mathfrak{E}} + \lambda H_I \quad (71)$$

where H depends only on the j and their generalized momenta, $H_{\mathfrak{E}}$ depends only on the ξ and its momenta, and H_I depends on both types of coordinates. We shall take the approximation, that H is the Hamiltonian of \mathfrak{S} when isolated, and that H_I alone models the interaction between \mathfrak{S} and \mathfrak{E} .

In principle, there can happy instances in which $\Psi(j, \xi)$ could be written as the product of two states functions:

$$\Psi(j, \xi) = \psi(j)\phi(\xi) \quad (72)$$

but this does not need to be the case. States that can be written in the form (72) are called *pure states* in the literature. States that are not pure are said to be *mixed*.

In order to illustrate this let us consider the case in which both the original system and its surroundings can be represented as two-level systems. If the isolated Hamiltonian H has eigenfunctions ψ_+ and ψ_- :

$$H\psi_{\pm} = E_{\pm}\psi_{\pm} \quad (73)$$

and if ϕ_{α} and ϕ_{β} are the eigenfunctions of H_e , *i. e.*

$$H_e\phi_{\alpha} = E_{\alpha}\phi_{\alpha} , \quad H_e\phi_{\beta} = E_{\beta}\phi_{\beta}, \quad (74)$$

then some examples of pure states are:

$$\begin{aligned} \frac{1}{\sqrt{2}}(\psi_+\phi_{\beta} + \psi_-\phi_{\beta}) &= \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)\phi_{\beta}, \quad \frac{1}{2}\psi_-\phi_{\beta} + \frac{\sqrt{3}}{2}\psi_-\phi_{\alpha} = \psi_-(\frac{1}{2}\phi_{\beta} + \frac{\sqrt{3}}{2}\phi_{\alpha}) \\ &\text{and} \\ \frac{1}{4}\psi_-\phi_{\beta} + \frac{\sqrt{3}}{4}\psi_-\phi_{\alpha} - \frac{\sqrt{3}}{4}\psi_-\phi_{\beta} - \frac{3}{4}\psi_+\phi_{\alpha} &= (\frac{1}{2}\psi_- - \frac{\sqrt{3}}{2}\psi_+)(\frac{1}{2}\psi_{\beta} + \frac{\sqrt{3}}{2}\phi_{\alpha}). \end{aligned}$$

On the other hand, as instances of mixed states, we can provide the following:

$$\frac{1}{\sqrt{2}}(\psi_-\phi_{\alpha} + \psi_+\phi_{\beta}), \quad \frac{1}{\sqrt{2}}(\psi_-\phi_{\beta} + \psi_+\phi_{\alpha}), \quad \text{and} \quad \frac{1}{\sqrt{3}}(\psi_+\phi_{\alpha} + \psi_+\phi_{\beta} + \psi_-\psi_{\alpha}). \quad (75)$$

The density matrix formalism was developed to treat systems that (like \mathfrak{U}) can present mixed states. The density matrix ρ allows us to calculate the expected value $\langle f \rangle$ of any observable $f(x, p_x)$ that depends only on the coordinates and momenta of \mathfrak{S} :

$$\langle f \rangle = \text{Tr}(f\rho). \quad (76)$$

The elements of the density matrix ρ of a state $\Psi(j, \xi)$ of are defined as:

$$\rho_{j,j'} = S_{\xi}\Psi^*(j, \xi)\Psi(j', \xi), \quad (77)$$

where S_{ξ} stands for the sum over the discrete ξ (if any) plus an integral over the continuous ξ (if any). In the case of our 2×2 -level system, expression (78) reduces to

$$\rho_{j,j'} = \Psi_{j,\alpha}^*\Psi_{j',\alpha} + \Psi_{j,\beta}^*\Psi_{j',\beta}, \quad j, j' = \pm. \quad (78)$$

The diagonal elements of density matrix, of the form $\rho_{j,j}$, are called *populations*, while the off-diagonal elements (*i. e.* the elements with $j \neq j'$) are known as *coherences*.

Suppose now that the 50-50 linear combinations

$$\psi_L = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \psi_R = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-) \quad (79)$$

are eigenfunctions of a macroscopic observable M , let us say:

$$M\psi_{L,R} = \mu_{L,R}\psi_{L,R}, \quad (80)$$

and take then the mixed state given by

$$\Psi = c_L\psi_L\phi_\alpha + c_R\psi_R\phi_\beta. \quad (81)$$

The density matrix associated with (81) is written as

$$\rho = \begin{pmatrix} |c_L|^2 & 0 \\ 0 & |c_R|^2 \end{pmatrix} \quad (82)$$

in the $\{\psi_L, \psi_R\}$ basis, as can be seen from (78), so that according to (76) the expected value of any observable f pertaining to \mathfrak{S} yields the value

$$\langle f \rangle = |c_L|^2 f_L + |c_R|^2 f_R, \quad (83)$$

where f_L and f_R are the expected values of f in the pure states

$$\Psi_L = \psi_L\phi_\alpha \quad \text{and} \quad \Psi_R = \psi_R\phi_\beta. \quad (84)$$

The point of this discussion is that the same result (83) is obtained if we make measurements on an ensemble of \mathfrak{U} systems all in state Ψ , or if the same measurements are made with an ensemble of \mathfrak{U} made up of a statistical mixture of the pure states Ψ_L and Ψ_R , in proportions $|c_L|^2$ and $|c_R|^2$. If it were to be held true that only ensembles of the type (81) could be prepared for \mathfrak{U} , then it could be argued that property M has a sharp value for each element of the ensemble, and that a measurement done on a particular element only removes our ignorance on its value for that particular system. Clearly, this opens the door for hidden variable theories. To put it succinctly: in this interpretation each one of the Schroedinger's cats in an ensemble of such felines would be either dead or alive, and never in superpositions composed of both dead and alive states. Only the behaviour of the ensemble would be quantal, its individual elements being essentially classical.

It is patent, on the other hand, that the pure state

$$\Psi_+ = \psi_+ \phi_\alpha \quad (85)$$

cannot be written as a mixed state of the form (81) and that its corresponding density matrix cannot be diagonal in the (L, R) basis, unlike (82). The impossibility of the simultaneous diagonalization of the density matrices of all possible states of a system \mathfrak{S} is then a strong evidence of the true quantal behaviour of such system, as opposed to the behaviour required by hidden variable theories. Thus, for a system S to be classical in any sense of the word, the coherences, *i. e.* the off-diagonal elements, must be absent from the density matrix for each one of its possible states. This conclusion is generally valid, even if we resorted to the most trivial case in order to illustrate it.¹

Decoherence can be defined as the decay of the off-diagonal elements in the density matrix as a result of the interaction of the system with its environment. Therefore decoherence allows a system to behave as quantal when isolated and as classical when the coupling with its environment is “sufficiently effective.” This is nowadays considered a plausible mechanism for the emergence of classical reality from a quantal substratum.

In most practical applications, the environment \mathfrak{E} has a very large number of degrees of freedom (say of the order of the Avogadro number) and not just one, as in the example we have used. Thus \mathfrak{U} is usually a thermodynamic system, so that the full toolbox of quantum statistical mechanics needs to be marshalled in order to describe its behaviour. In this case the interaction between \mathfrak{S} and \mathfrak{E} (interaction known as quantum dissipation in this context) involves the relaxation of the thermodynamical variables of \mathfrak{U} towards thermal equilibrium, and not only decoherence.

Various models have been proposed over the years for the environment (or *bath*) but one of first and most successful is the *spin-boson Hamiltonian* approach, in which $H_{\mathfrak{E}}$ is taken as a collection of harmonic oscillators with various frequencies and the interaction term H_I is linear both in the j and in the ξ coordinates. One important result from this approach is that a two-level system \mathfrak{S} will describe damped oscillations between the localized states $|R\rangle$ and $|L\rangle$. Depending on the frequency distribution of the environment, \mathfrak{S} may be localized at $T = 0^\circ\text{K}$ (the overdamped case, known as “subohmic”), it may present critical damping (the “ohmic case”) or it may undergo underdamped coherent oscillations (the “superohmic case.”) The last one of these three instances is the most interesting for

the present discussion, as it allows the observation of MQC before the complete relaxation of the system. The possibility of experimental MQC in the superohmic case depends in the interplay between a *decoherence time* defined only by the bath parameters, and the period of the Leggett-Caldeira oscillation for system \mathfrak{S} .

VI. THE SCALES OF MQC

b (nm)	E_0 ($\times 10^{-26}$ J)	E_1 ($\times 10^{-26}$ J)	ΔE ($\times 10^{-28}$ J)	τ (μs)
100.00000	5.3753895	5.4382093	6.3	1.0
116.65290	5.3899569	5.4246062	3.5	2.9
136.07900	5.3987829	5.4160961	1.7	3.8
158.74011	5.4036276	5.4113353	0.77	8.6
185.17494	5.4059909	5.4089897	0.30	22.0
216.01195	5.4069931	5.4079902	0.10	66.0
251.98421	5.4073539	5.4076298	2.7×10^{-2}	240

TABLE I. Period τ increases exponentially as the barrier width is augmented. $a = 1.0 \mu m$, $k = 2 \times 10^{-20} J$. This table, as well as all figures, was generated with Matlab [®] R2012a.

Let us start by fixing the width of the lateral wells at:

$$a = 1 \mu m, \quad (86)$$

a value typical of contemporary lithographic circuitry, and take m to be the rest mass of an electron:

$$m = m_e = 9.1 \times 10^{-31} kg. \quad (87)$$

With this, B takes the value:

$$B = 0.6 \times 10^{-25} J = 0.36 \mu eV, \quad (88)$$

and T_B is fixed at:

$$T_B \approx 1.1 \text{ mK}. \quad (89)$$

From equation (25), that gives the fundamental frequency of the Caldeira-Leggett oscillations, we get the corresponding period

$$\tau = \frac{2\pi\hbar}{E_1 - E_0}. \quad (90)$$

A global lower bound for this period is found from expressions (12) and (13):

$$\tau > \frac{2\pi\hbar}{B} = \frac{4ma^2}{\pi\hbar}. \quad (91)$$

For values (86) and (87) this gives

$$\tau > 11\text{ns}. \quad (92)$$

From table I (obtained through computer assisted numerical analysis) we get that as we sweep the barrier width from 0.2 to 0.5 μm the period of the Leggett -Caldeira oscillations for our square double well increases from 1.0 to 240 μs . Based on general considerations it has been estimated¹ that, for all practical purposes, MQC is lost if the period of the Leggett-Caldiera oscillation is of the order $\tau \gtrsim 100\mu\text{s}$. Thus, the last row of the table corresponds to a localized system. All the other tabulated values could in principle correspond to observable MQC.

A. Some of the many things we have left out

MQC experiments are carried out in superconducting quantum interference devices (SQUIDS) with low capacitance tunneling Josephson junctions,^{1,4} and the relevant coordinate (*i. e.* the analogous of coordinate x) is not of a geometric character (like a position) but is in most cases the phase difference between the states functions of the electrons in a Cooper pair (so that m is not really the mass of the electron.) Thus our toy model is in reality a simplification of a mechanical analogy used to discuss experimental MQC.

VII. CONCLUSIONS

Contemporary quantum mechanics, both experimental and theoretical, provides examples of basic concepts and techniques such as: tunneling, stationary states, two-level systems, perturbation theory, the density matrix and the WKB approximation. Classroom presentations of current areas of research, such as MQC, help to improve the understanding of

quantum physics at university level, as they connect the simplified textbook models with the actual state of the field, and thus with the future professional activity of the student. Moreover, MQC illustrates in a beautiful way the interplay between theory and experiment, and between concepts and techniques arising in different areas of quantum physics.

We believe to have achieved in the present paper a level of exposition that makes it both clear and interesting for senior university students and recent graduates. In order to do so, we had to glide over the more technical aspects of experimental MQC and the intricate relation between MQC and the epistemology and the philosophy of physics. We hope that the present paper will encourage the interested reader to delve further into this facets of contemporary research.

VIII. APPENDIX

Consider condition (8) for the ground level ($n = 0$), that is:

$$E_0 \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} = (k - E_0) \tanh^2 b \frac{\sqrt{2m(k - E_0)}}{\hbar} \quad (93)$$

We will now establish a lower bound for E_0 starting from (93), but we have to take some precautions in doing so because E_0 depends implicitly on b . In order to proceed, note that

$$\forall b \in (0, \infty), \quad \frac{\sqrt{2m(k - E_0)}}{\hbar} < \frac{\sqrt{2m(k - B/4)}}{\hbar} \quad (94)$$

so that

$$\forall b \in (0, \infty), \quad \tanh^2 b \frac{\sqrt{2m(k - E_0)}}{\hbar} > \tanh^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} \quad (95)$$

The dependence of the rhs of inequality (95) is explicit, so that the usual procedures of calculus can be applied. In particular as we now from elementary theorems that the limit

$$\lim_{b \rightarrow \infty} \tanh^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} = 1 \quad (96)$$

holds true, we can affirm that: for given $\delta > 0$ there exists a $b_0(\delta)$ such that any $b > b_0(\delta)$

$$\tanh^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} > 1 - \frac{\delta}{2k} \quad (97)$$

From (93) (95) and (97) we deduce that for any b above a certain value $b_0(\delta)$, the ground energy of U_b satisfies:

$$E_0 \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} > (k - E_0) \left(1 - \frac{\delta}{2k}\right) \quad (98)$$

Turning our attention to the condition for E_1 , *i. e.*

$$E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} = (k - E_1) \coth^2 b \frac{\sqrt{2m(k - E_1)}}{\hbar} \quad (99)$$

we now find an upper bound for E_1 , by noting that, because of (94) and the known properties of the hyperbolic functions, the inequality

$$\coth^2 b \frac{\sqrt{2m(k - E_1)}}{\hbar} < \coth^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} \quad (100)$$

is verified for all strictly positive b . Furthermore,

$$\lim_{b \rightarrow \infty} \coth^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} = 1 \quad (101)$$

so that for every $\delta > 0$ there exist a $b_1(\delta)$ such that, if $b > b_1(\delta)$, then inequality

$$\coth^2 b \frac{\sqrt{2m(k - B/4)}}{\hbar} < 1 + \frac{\delta}{2k} \quad (102)$$

is satisfied for all strictly positive b . And from (99) and (102) we get that, for all b above a certain threshold value $b_1(\delta)$, the inequality

$$E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} < (k - E_1) \left(1 + \frac{\delta}{2k}\right) \quad (103)$$

is satisfied.

Taking both (98) and (103) into consideration, we have that for every $\delta > 0$ there exists a number $b(\delta) = \max\{b_0(\delta)b_1(\delta)\}$ such that for any $b > b(\delta)$ the inequality

$$E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} - E_0 \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} < (E_0 - E_1) + \delta \left(1 - \frac{E_0 + E_1}{2k}\right) \quad (104)$$

is satisfied. Now, it is not difficult to see that

$$w(E) = E \cot^2 a \frac{\sqrt{2mE}}{\hbar} \quad (105)$$

is a monotonically increasing function of E in the range $B/4 < E < B$, so that

$$0 < E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} - E_0 \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} \quad (106)$$

and in the other hand, we deduce

$$(E_0 - E_1) + \delta \left(1 - \frac{E_0 + E_1}{2k}\right) < \left(1 - \frac{E_0 + E_1}{2k}\right) \delta < \delta \quad (107)$$

from and . From (104), (106) and (107) we get:

$$0 < E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} - E_0 \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} < \delta \quad (108)$$

$$E_1 \cot^2 a \frac{\sqrt{2mE_1}}{\hbar} > k - E_1 \quad (109)$$

Finally, we notice that, as

$$v(E) = \cot^2 a \frac{\sqrt{2mE}}{\hbar} \quad (110)$$

is a monotonically increasing function of E in the range $B/4 < E < B$, then

$$(E_1 - E_0) \cot^2 a \frac{\sqrt{2mE_0}}{\hbar} < \delta \quad (111)$$

Now we just need to find a lower bound on $\cot^2 a \frac{\sqrt{2mE_0}}{\hbar}$. This is obtained by turning back to condition (93) from which we get

$$\cot^2 a \frac{\sqrt{2mE_0}}{\hbar} < \frac{k - B/4}{B} \quad (112)$$

Finally, from (111) and (112) we arrive at

$$E_1 - E_0 < \delta \frac{B}{k - B/4} \quad (113)$$

Let us stress that k and B are independent of b . In this manner, we have arrived at the following lemma:

For each strictly positive real number δ there exists a

$$b'(\delta) = b\left(\delta \frac{k - B/4}{B}\right) \quad (114)$$

such that for any $b > b'(\delta)$ the gap between the ground and first excited levels of U_b is less than δ , that is, such that:

$$E_1 - E_0 < \delta. \quad (115)$$

And this is what we set out to prove in this appendix.

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- ¹ A J Leggett “Macroscopic Quantum Systems and the Quantum Theory of Measurement,” *Prog. Theor. Phys. Supplement* **69** 80-100 (1980)
- ² Y Nakamura, Yu A Pashkin and J S Tsai “Coherent control of macroscopic quantum states in a single-Cooper-pair box” *Nature* **398** 786-788 (1999)
- ³ Y Makhlin, G Schön and A Shnirman “Quantum-state engineering with Josephson-junction devices” *Rev. Mod. Phys.* **73**, 2, 357-400 (2001)
- ⁴ J R Friedman, V Patel, W Chen, S K Tolpygo and E Lukens “Quantum superposition of distinct macroscopic states” *Nature* **406** 43-46 (2000)
- ⁵ G Wendin and V S Shumeiko “Quantum bits with Josephson junctions (review article)” *Low Temp. Phys.* **33** (9) 724-744 (2007)
- ⁶ A O Caldeira and A J Leggett, “Influence of dissipation on quantum tunneling in macroscopic systems”, *Phys. Rev. Lett.* **46** (4) 211-214 (1981)
- ⁷ A J Leggett and A Garg “Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?” *Phys. Rev. Lett.* **54** (9) 857-860 (1985)
- ⁸ A J Leggett, S Chakravarty, A T Dorsey, M P A Fisher, A Garg and W Zwerger “Dynamics of the dissipative two-state system” *Rev. Mod. Phys.* **59** 1-85 (1987)
- ⁹ C D Tesche “Can a noninvasive measurement of magnetic flux be performed with superconducting circuits?” *Phys. Rev. Letters* **64** (20) 2358-2361 (1990)
- ¹⁰ P Carelli, M G Castellano, F Chiarello, C Cosmelli, R Leoni and G. Torrioli “SQUID Systems for Macroscopic Quantum Coherence and Quantum Computing,” *IEEE Trans. Appl. Supercond.* **11** (1) 210-214 (2001)
- ¹¹ V E Manucharyan, J Koch, L I Glazman and M H Devoret “Fluxonium: Single cooper-pair circuit free of charge offsets,” *Science* **326** (5949) 113-116 (2009)
- ¹² V. E. Manucharyan, J. Koch, M. Brink, L. I. Glazman and M. H. Devoret “Coherent oscillations between classically separable quantum states of a superconducting loop” [arXiv:0910.3039](https://arxiv.org/abs/0910.3039)
- ¹³ D Bohm *Quantum Theory* (Dover, Mineola, 1979) Ch. 12 §1
- ¹⁴ L D Landau and E M Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (Addison-Wesley, Reading, 1965) Ch. I, §6
- ¹⁵ L D Landau and E. M. Lifshitz, *Op. Cit.* Ch. III, §21
- ¹⁶ L D Landau and E M Lifshitz *Op. Cit.* Ch. VII, §50, solved example 3

- ¹⁷ V Jelic and F Marsiglio “The double-well potential in quantum mechanics: a simple, numerically exact formulation” *Eur. J. Phys.* **33** (6) 1651-1660 (2012)
- ¹⁸ R Muñoz-Vega, A García-Quiroz, E López-Chávez and E Salinas-Hernández “Spontaneous symmetry breakdown in non-relativistic quantum mechanics” *Am. J. Phys.* **80** (10) 891-897 (2012)
- ¹⁹ R Feynman, R B Leighton and M Sands *The Feynman Lectures on Physics* 1st Edition (Addison-Wesley, Palo Alto, 1965) Vol. III, Chapters 9, 10,11 and 12
- ²⁰ C Cohen-Tannoudji, B Diu and F Laloe *Quantum Mechanics* Vol. I, Chapter IV (Wiley-Interscience, Hoboken,1992)
- ²¹ L D Landau and E M Lifshitz *Op. Cit.* Ch. VI §40
- ²² L D Landau and E M Lifshitz *Op.Cit.* Ch. VI §40.
- ²³ R Fitzpatrick “Quantum Mechanics: A graduate level course”
<http://farside.ph.utexas.edu/teaching/qm/lectures/lectures.html> Ch.V, Sec. 9
- ²⁴ R Fitzpatrick *Op. Cit.* Ch.V, Sec. 10
- ²⁵ G Esposito, G Marmo and G Sudarshan *From Classical to Quantum Mechanics* (CUP, Cambridge, 2004) Pages 269-274.
- ²⁶ L D Landau and E M Lifshitz *Op. Cit.* Ch.II, §14